ON CONSTRUCTING PERIODIC SOLUTIONS OF QUASI-LINEAR NON-SELF-CONTAINED SYSTEMS WITH ONE DEGREE OF FREEDOM, WHEN AMPLITUDE EQUATIONS POSSESS MULTIPLE ROOTS

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Construction of periodic solutions of quasilinear non-self-contained systems with one degree of freedom, was investigated in [1 and 2]. In [1] the case of simple roots of amplitude equations was considered together with the case of a double root when the solution could be expanded into a series in integral powers of μ . In [2] the case of a double root is investigated in more detail including expansions of solutions into series in $\mu^{1/2}$. In the present paper, the case of arbitrary multiple roots for non-self-contained systems is reduced to the corresponding case for self-contained systems, which simplifies computations.

1. Let us consider a quasi-linear non-self-contained system

$$\ddot{x} + m^2 x = f(t) + \mu F(t, x, x^{\cdot}, \mu) \tag{1.1}$$

Function $F(t, x, x^{\cdot}, \mu)$ is analytic in x, $x^{\cdot}$ and μ in some region of variation of x and $x^{\cdot}$, with $0 \le \mu < \mu_0$ (μ is a small positive parameter). Also, $F(t, x, x^{\cdot}, \mu)$ and $f(t)$ are continuous periodic functions of t with the period 2π and harmonics of the mth order where m is an integer, are absent from $f(t)$

When $\mu = 0$, the basic system has a general solution

$$x_0(t) = \varphi(t) + A_0 \cos mt + B_0 m^{-1} \sin mt \tag{1.2}$$

periodic in t and dependent on two arbitrary constants A_0 and B_0

We shall seek periodic solutions of (1.1) using Poincaré's method. Assume the initial conditions

$$x(0) = \varphi(0) + A_0 + \beta, \qquad x^{\cdot}(0) = \varphi^{\cdot}(0) + B_0 + \gamma \tag{1.3}$$

Solution of (1.1) is an analytic function of $A_0 + \beta$, $B_0 + \gamma$ and μ , consequently, it can be written in the form

$$x(t) = \varphi(t) + (A_0 + \beta)\cos mt + \frac{B_0 + \gamma}{m}\sin mt + \sum_{n=1}^{\infty} C_n(t, A_0 + \beta, B_0 + \gamma)\mu^n \tag{1.4}$$

or, on expanding

$$x(t) = \varphi(t) + (A_0 + \beta)\cos mt + \frac{B_0 + \gamma}{m}\sin mt +$$
$$+ \sum_{n=1}^{\infty}\left[C_n(t) + \frac{\partial C_n(t)}{\partial A_0}\beta + \frac{\partial C_n(t)}{\partial B_0}\gamma + \frac{1}{2}\frac{\partial^2 C_n(t)}{\partial A_0^2}\beta^2 + \ldots\right]\mu^n \qquad (1.5)$$

Functions $C_n(t)$ are given by

$$C_n(t) = \frac{1}{m}\int_0^t H_n(t')\sin m(t - t')dt' \qquad (1.6)$$

where $H_n(t)$ are

$$H_n(t) = \frac{1}{(n-1)!}\left(\frac{d^{n-1}F}{d\mu^{n-1}}\right)_{\beta=\gamma=\mu=0}$$

$dF/d\mu$ is a complete partial differential of $F(t, x, x^{\cdot}, \mu)$ with respect to μ. Values of first three $H_n(t)$ are given in their expanded form in [1].

Conditions of periodicity which, with initial conditions taken into account, can be given as

$$x(2\pi) = \varphi(0) + A_0 + \beta, \qquad x^{\cdot}(2\pi) = \varphi^{\cdot}(0) + B_0 + \gamma \qquad (1.7)$$

yield, together with (1.4), the following relationships:

$$\sum_{n=1}^{\infty} C_n(2\pi,\ A_0 + \beta,\ B_0 + \gamma)\mu^n = 0, \qquad \sum_{n=1}^{\infty} C_n^{\cdot}(2\pi,\ A_0 + \beta,\ B_0 + \gamma)\mu^n = 0 \qquad (1.8)$$

from which we can find the following equations of fundamental amplitudes A_0 and B_0

$$C_1(2\pi, A_0, B_0) = 0, \qquad C_1^{\cdot}(2\pi, A_0, B_0) = 0 \qquad (1.9)$$

together with equations giving the parameters β and γ as implicit functions of μ. Regrouping the terms in these equations so as to obtain homogeneous polynomials, we obtain

$$\frac{\partial C_1}{\partial A_0}\beta + \frac{\partial C_1}{\partial B_0}\gamma + C_2\mu + \frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2}\beta^2 + \frac{\partial^2 C_1}{\partial A_0 \partial B_0}\beta\gamma + \frac{1}{2}\frac{\partial^2 C_1}{\partial B_0^2}\gamma^2 +$$
$$+ \frac{\partial C_2}{\partial A_0}\beta\mu + \frac{\partial C_2}{\partial B_0}\gamma\mu + C_3\mu^2 + \ldots = 0 \qquad (1.10)$$

and analogously

$$\frac{\partial C_1^{\cdot}}{\partial A_0}\beta + \frac{\partial C_1^{\cdot}}{\partial B_0}\gamma + C_2^{\cdot}\mu + \frac{1}{2}\frac{\partial^2 C_1^{\cdot}}{\partial A_0^2}\beta^2 + \frac{\partial^2 C_1^{\cdot}}{\partial A_0 \partial B_0}\beta\gamma + \frac{1}{2}\frac{\partial^2 C_1^{\cdot}}{\partial B_0^2}\gamma^2 +$$
$$+ \frac{\partial C_2^{\cdot}}{\partial A_0}\beta\mu + \frac{\partial C_2^{\cdot}}{\partial B_0}\gamma\mu + C_3^{\cdot}\mu^2 + \cdots = 0 \qquad (1.11)$$

In case of simple roots of amplitude equations (1.9), functional determinant

$$D_1 = \frac{\partial C_1}{\partial A_0}\frac{\partial C_1^{\cdot}}{\partial B_0} - \frac{\partial C_1}{\partial B_0}\frac{\partial C_1^{\cdot}}{\partial A_0} \qquad (1.12)$$

is not equal to zero, from which it follows that β and γ can be represented by series in integral powers of μ

$$\beta = \sum_{n=1}^{\infty} A_n\mu^n, \qquad \gamma = \sum_{n=1}^{\infty} B_n\mu^n \qquad (1.13)$$

Coefficients A_n and B_n of these series can be determined consecutively from an infinite system of pairs of linear equations possessing a common determinant equal to D_1. Equations for the first three coefficients A_n and B_n are given in [1].

2. We shall now consider the case of multiple roots of (1.9) in more detail. The necessary condition for multiple roots to exist, is

$$D_1 = 0 \qquad (2.1)$$

Each of the equations of (1.9) defines some curve on the plane of amplitudes A_0 and B_0. Points of intersection of these curves are the roots of our equations. Double roots correspond to double points of intersection, triple roots to triple points, etc., etc.

We shall first investigate the points, at which simple tangency of two curves occurs. Differentiating (1.9) with respect to A_0 and assuming that B_0 is an implicit function of A_0, we obtain

$$\frac{\partial C_1}{\partial A_0}+\frac{\partial C_1}{\partial B_0}\frac{dB_0}{dA_0}=0, \qquad \frac{\partial C_1^{\cdot}}{\partial A_0}+\frac{\partial C_1^{\cdot}}{\partial B_0}\frac{dB_0}{dA_0}=0 \tag{2.2}$$

Since at the points of contact direction cosines of the tangents coincide, it follows that at these points $D_1 = 0$.

Second differentiation with respect to A_0, yields

$$\begin{aligned}\frac{\partial^2 C_1}{\partial A_0^2}+2\frac{\partial^2 C_1}{\partial A_0\partial B_0}\frac{dB_0}{dA_0}+\frac{\partial^2 C_1}{\partial B_0^2}\left(\frac{dB_0}{dA_0}\right)^2+\frac{\partial C_1}{\partial B_0}\frac{d^2B_0}{dA_0^2}=0\\ \frac{\partial^2 C_1^{\cdot}}{\partial A_0^2}+2\frac{\partial^2 C_1^{\cdot}}{\partial A_0\partial B_0}\frac{dB_0}{dA_0}+\frac{\partial^2 C_1^{\cdot}}{\partial B_0^2}\left(\frac{dB_0}{dA_0}\right)^2+\frac{\partial C_1^{\cdot}}{\partial B_0}\frac{d^2B_0}{dA_0^2}=0\end{aligned} \tag{2.3}$$

Points of contact at which dB_0/dA_0 and d^2B_0/dA_0^2 coincide, represent triple roots of Equations (1.9). At these points

$$D_2=\begin{vmatrix}\frac{\partial^2 C_1}{\partial A_0^2}+2\frac{\partial^2 C_1}{\partial A_0\partial B_0}\frac{dB_0}{dA_0}+\frac{\partial^2 C_1}{\partial B_0^2}\left(\frac{dB_0}{dA_0}\right)^2 & \frac{\partial C_1}{\partial B_0}\\ \frac{\partial^2 C_1}{\partial A_0^2}+2\frac{\partial^2 C_1^{\cdot}}{\partial A_0\partial B_0}\frac{dB_0}{dA_0}+\frac{\partial^2 C_1^{\cdot}}{\partial B_0^2}\left(\frac{dB_0}{dA_0}\right)^2 & \frac{\partial C_1^{\cdot}}{\partial B_0}\end{vmatrix}=0 \tag{2.4}$$

which is the necessary condition of existence of triple roots of (1.9), and which supplements (2.1). Condition (2.4) can be represented in various ways, which depend on the form taken by the derivative dB_0/dA_0. We can obtain dB_0/dA_0 either from the first or from the second equation of (2.2), or from both. At the same time we shall assume, that in the corresponding cases either one or both of the following inequalities hold

$$\frac{\partial C_1}{\partial B_0}\neq 0, \qquad \frac{\partial C_1^{\cdot}}{\partial B_0}\neq 0$$

Condition of triplicity of the roots of amplitude equations can be obtained in another form, if Equations (1.9) are differentiated twice with respect to B_0, with A_0 assumed to be an implicit function of B_0. We shall have then, in place of (2.4) a condition, in which differentiation with respect to B_0 replaces the differentiation with respect to A_0, and vice-versa, and dA_0/dB_0 replaces dB_0/dA_0.

Therefore, provided that at least one of the first partial derivatives of C_1 and $C_1^{\cdot}$ with respect to A_0 and B_0 is different from zero, conditions of existence of double roots of amplitude equations (1.9) will be: fulfilment of condition (2.1) and nonfulfilment of condition (2.4), or of another condition possessing at least one analogous aspect.

Let us consider some particular cases. Assume, that one of the curves defined by (1.9) has double branches, e.g.

$$C_1(A_0, B_0) = f_1^2(A_0, B_0)\, f_2(A_0, B_0) = 0 \tag{2.5}$$

Then, at any point of intersection of the double branch with the curve given by $C_1^{\cdot}(A_0, B_0) = 0$, corresponds to multiple roots of the system. At the same time condition (2.1) always holds. In the present case however, (2.1) is not a condition of tangency of the curves. The latter condition is expressed in the form

$$D_1^* = \frac{\partial f_1}{\partial A_0}\frac{\partial C_1^{\cdot}}{\partial B_0}-\frac{\partial f_1}{\partial B_0}\frac{\partial C_1^{\cdot}}{\partial A_0}=0 \tag{2.6}$$

Consequently, lack of fulfilment of (2.6) is the condition for the roots of (1.9) to be double for the given double branch.

Let us consider another particular case. Let our amplitude equations have the form

$$C_1(A_0) = 0, \qquad C_1^{\cdot}(B_0) = 0 \tag{2.7}$$

This represents two orthogonal straight lines parallel to the coordinate axes on the A_0B_0 plane. They intersect at a double point, provided one of the equations (2.7), say the first one, has double roots

$$\frac{\partial C_1}{\partial A_0} = 0, \qquad \frac{\partial^2 C_1}{\partial A_0^2} \neq 0, \qquad \frac{\partial C_1^{\cdot}}{\partial B_0} \neq 0 \tag{2.8}$$

The fulfilment of (2.1) follows from these conditions, and they are the conditions of existence of double roots.

3. Parameters β and γ can be found from (1.8) with (1.9), or from expanded equations (1.10) and (1.11). Let us transform these equations, assuming that

$$\frac{\partial C_1}{\partial B_0} \neq 0 \tag{3.1}$$

Second relation of (1.8) gives, together with (1.9), $B_0 + \gamma$ as an analytic function of $A_0 + \beta$ and μ

$$B_0 + \gamma = f^*(A_0 + \beta, \mu)$$

Inserting this into the first relation of (1.8) and dividing by μ, we obtain

$$\sum_{n=1}^{\infty} C_n[A_0 + \beta, f^*(A_0 + \beta, \mu)]\mu^{n-1} = \sum_{n=1}^{\infty} Q_n(A_0 + \beta)\mu^{n-1} = 0 \tag{3.2}$$

We easily see, that $Q_1 = C_1(A_0, B_0) = 0$. Regrouping the terms so as to obtain homogeneous polynomials, we obtain (3.2) in an expanded form

$$\Phi(\beta, \mu) = \frac{dQ_1}{dA_0}\beta + Q_2\mu + \frac{1}{2}\frac{d^2Q_1}{dA_0^2}\beta^2 + \frac{dQ_2}{dA_0}\beta\mu + Q_3\mu^2 +$$
$$+ \frac{1}{6}\frac{d^3Q_1}{dA_0^3}\beta^3 + \frac{1}{2}\frac{d^2Q_2}{dA_0^2}\beta^2\mu + \frac{dQ_3}{dA_0}\beta\mu^2 + Q_4\mu^3 + \cdots = 0 \tag{3.3}$$

where

$$\frac{dQ_n}{dA_0} = \frac{\partial Q_n}{\partial A_0} + \frac{\partial Q_n}{\partial B_0}\frac{dB_0}{dA_0} \tag{3.4}$$

First two derivatives of Q_1 with respect to A_0 are

$$\frac{dQ_1}{dA_0} = D_1\left(\frac{\partial C_1^{\cdot}}{\partial B_0}\right)^{-1}, \qquad \frac{d^2Q_1}{dA_0^2} = D_2\left(\frac{\partial C_1^{\cdot}}{\partial B_0}\right)^{-1} \tag{3.5}$$

To obtain the coefficients of Q_n $(n = 2, 3, \ldots)$ we shall first find γ from (1.11) as a double series in β and μ. This is always possible by virtue of (3.1). Regrouping the terms of this expansion in the same manner as (3.3), we find

$$\gamma = \frac{dP_1}{dA_0}\beta + P_2\mu + \frac{1}{2}\frac{d^2P_1}{dA_0^2}\beta^2 + \frac{dP_2}{dA_0}\beta\mu + P_3\mu^2 +$$
$$+ \frac{1}{6}\frac{d^3P_1}{dA_0^3}\beta^3 + \frac{1}{2}\frac{d^2P_2}{dA_0^2}\beta^2\mu + \frac{dP_3}{dA_0}\beta\mu^2 + P_4\mu^3 + \ldots \tag{3.6}$$

Let us first find dP_1/dA_0 by inserting γ into (1.11) and collecting similar terms. Expansion obtained in terms β and μ is identically equal to zero. Equating the coefficient of β to zero, yields

$$\frac{dP_1}{dA_0}\frac{\partial C_1^{\cdot}}{\partial B_0} + \frac{\partial C_1^{\cdot}}{\partial A_0} = 0$$

from which we obtain

$$\frac{dP_1}{dA_0} = -\frac{\partial C_1^{\cdot}}{\partial A_0}\left(\frac{\partial C_1^{\cdot}}{\partial B_0}\right)^{-1} = \left(\frac{dB_0}{dA_0}\right)_* \tag{3.7}$$

where an asterisk denotes a magnitude taken at the tangent point of $C_1(A_0, B_0) = 0$ and $C_1^{\cdot}(A_0, B_0) = 0$.

To determine the coefficients of P_n $(n = 2, 3, \ldots)$ we proceed in a similar manner as for finding dP_1/dA_0, and we obtain

$$P_2 \frac{\partial C_1^{\cdot}}{\partial B_0} + C_2^{\cdot} = 0$$

$$P_3 \frac{\partial C_1}{\partial B_0} + \frac{1}{2} P_2^2 \frac{\partial^2 C_1^{\cdot}}{\partial B_0^2} + P_2 \frac{\partial C_2^{\cdot}}{\partial B_0} + C_3^{\cdot} = 0 \qquad (3.8)$$

$$P_4 \frac{\partial C_1^{\cdot}}{\partial B_0} + P_3 \left(P_2 \frac{\partial^2 C_1^{\cdot}}{\partial B_0^2} + \frac{\partial C_2^{\cdot}}{\partial B_0}\right) + \frac{1}{6} P_2^3 \frac{\partial^3 C_1^{\cdot}}{\partial B_0^3} + \frac{1}{2} P_2^2 \frac{\partial^2 C_2^{\cdot}}{\partial B_0^2} + P_2 \frac{\partial C_3^{\cdot}}{\partial B_0} + C_4^{\cdot} = 0$$

Inserting γ from (3.6) into (1.10), regrouping the terms and comparing the coefficients in the resulting equation with those of (3.3), we find

$$Q_2 = P_2 \frac{\partial C_1}{\partial B_0} + C_2, \quad Q_3 = P_3 \frac{\partial C_1}{\partial B_0} + \frac{1}{2} P_2^2 \frac{\partial^2 C_1}{\partial B_0^2} + P_2 \frac{\partial C_2}{\partial B_0} + C_3 \qquad (3.9)$$

$$Q_4 = P_4 \frac{\partial C_1}{\partial B_0} + P_3 \left(P_2 \frac{\partial^2 C_1}{\partial B_0^2} + \frac{\partial C_2}{\partial B_0}\right) + \frac{1}{6} P_2^3 \frac{\partial^3 C_1}{\partial B_0^3} + \frac{1}{2} P_2^2 \frac{\partial^2 C_2}{\partial B_0^2} + P_2 \frac{\partial C_3}{\partial B_0} + C_4$$

Introducing now

$$\Delta_1 = \frac{\partial C_1}{\partial B_0} C_2^{\cdot} - \frac{\partial C_1^{\cdot}}{\partial B_0} C_2, \qquad \Delta_2 = \frac{\partial C_1^{\cdot}}{\partial A_0} C_2 - \frac{\partial C_1}{\partial A_0} C_2^{\cdot} \qquad (3.10)$$

and using Formulas (3.8) for the coefficients of P_n, we finally have

$$Q_2 = - \Delta_1 \left(\frac{\partial C_1^{\cdot}}{\partial B_0}\right)^{-1} \qquad (3.11)$$

$$Q_3 = \left(\frac{\partial C_1^{\cdot}}{\partial B_0}\right)^{-3} \left[\frac{1}{2}\left(\frac{\partial^2 C_1}{\partial B_0^2} \frac{\partial C_1^{\cdot}}{\partial B_0} - \frac{\partial^2 C_1^{\cdot}}{\partial B_0^2} \frac{\partial C_1}{\partial B_0}\right) C_2^{\cdot 2} - \right.$$

$$\left. - \left(\frac{\partial C_2}{\partial B_0} \frac{\partial C_1^{\cdot}}{\partial B_0} - \frac{\partial C_2^{\cdot}}{\partial B_0} \frac{\partial C_1}{\partial B_0}\right) \frac{\partial C_1^{\cdot}}{\partial B_0} C_2^{\cdot} + \left(\frac{\partial C_1^{\cdot}}{\partial B_0} C_3 - \frac{\partial C_1}{\partial B_0} C_3^{\cdot}\right)\left(\frac{\partial C_1^{\cdot}}{\partial B_0}\right)^2\right] \qquad (3.12)$$

Coefficients Q_4 can be found directly from (3.9) and (3.8) rather than by means of a general formula.

Equation (3.3) possesses a structure similar to that of a corresponding equation for the case of self-contained system with one degree of freedom. Therefore the analysis of various cases of double and triple roots of amplitude equation of the self-contained system [3] can be applied in its entirety to non-self-contained systems.

Let us put $\mu = 0$ in the left-hand side of (3.3). This results in

$$\Phi(\beta, 0) = \frac{dQ_1}{dA_0} \beta + \frac{1}{2} \frac{d^2 Q_1}{dA_0^2} \beta^2 + \frac{1}{6} \frac{d^3 Q_1}{dA_0^3} \beta^3 + \ldots \qquad (3.13)$$

By the Weierstrass theorem for implicit functions, number of branches of the parameter β given by (3.3) is equal to the smallest exponent in the expansion of $\Phi(\beta, 0)$ in β which, in turn, is equal to the multiplicity of the roots of amplitude equations. If the multiplicity of roots is equal to r, then all of r branches of β are given by convergent series in integral powers of $\mu^{1/k}$, where k may be equal to any integer between 1 and r inclusive. At the same time, these roots of amplitude equations may possess expansions of β in various fractional powers of μ, but the sum of distinct k cannot exceed r.

Hence, β and γ are given by

$$\beta = \sum_{n=1}^{\infty} A_{n/k} \mu^{n/k}, \qquad \gamma = \sum_{n=1}^{\infty} B_{n/k} \mu^{n/k} \qquad (3.14)$$

Form and the coefficients of the series for β are found from (3.3), taking into account the multiplicity of roots of amplitude equations. For example, in the case of double roots and $Q_2 \neq 0$, we have $k = 2$. Coefficient $A_{\frac{1}{2}}$ is found from the quadratic equation

$$\frac{1}{2}\frac{d^2Q_1}{dA_0^2}A_{1/2}{}^2+Q_2=0 \quad \text{etc.} \tag{3.15}$$

Form of the series for γ is the same as of that for β. To find the coefficients $B_{n/k}$, the value of β from (3.14) should be inserted into (3.6) and all its terms expanded in powers of $\mu^{1/k}$. Then, the corresponding coefficients of this series and of the series given by the second formula of (3.14) are equated, giving for $k = 2$,

$$B_{1/2}=\frac{dP_1}{dA_0}A_{1/2}, \qquad B_1=\frac{dP_1}{dA_0}A_1+\frac{1}{2}\frac{d^2P_1}{dA_0^2}A_{1/2}{}^2+P_2$$

$$B_{3/2}=\frac{dP_1}{dA_0}A_{3/2}+\frac{d^2P_1}{dA_0^2}A_{1/2}A_1+\frac{1}{6}\frac{d^3P_1}{dA_0^3}A_{1/2}{}^3+\frac{dP_2}{dA_0}A_{1/2}$$

$$B_2=\frac{dP_1}{dA_0}A_2+\frac{d^2P_1}{dA_0^2}\left(A_{1/2}A_{3/2}+\frac{1}{2}A_1^2\right)+\frac{1}{2}\frac{d^3P_1}{dA_0^3}A_{1/2}{}^2A_1+$$
$$+\frac{dP_2}{dA_0}A_1+\frac{1}{2}\frac{d^2P_2}{dA_0^2}A_{1/2}{}^2+P_3 \tag{3.16}$$

and for $k = 3$

$$B_{1/3}=\frac{dP_1}{dA_0}A_{1/3}, \qquad B_{2/3}=\frac{dP_1}{dA_0}A_{2/3}+\frac{1}{2}\frac{d^2P_1}{dA_0^2}A_{1/3}{}^2$$

$$B_1=\frac{dP_1}{dA_0}A_1+\frac{d^2P_1}{dA_0^2}A_{1/3}A_{2/3}+\frac{1}{6}\frac{d^3P_1}{dA_0^3}A_{1/3}{}^3+P_2$$

$$B_{4/3}=\frac{dP_1}{dA_0}A_{4/3}+\frac{d^2P_1}{dA_0^2}\left(A_{1/3}A_1+\frac{1}{2}A_{2/3}{}^2\right)+\frac{1}{2}\frac{d^3P_1}{dA_0^3}A_{1/3}{}^2A_{2/3}+\frac{dP_2}{dA_0}A_{1/3} \tag{3.17}$$

If the inequality $\partial C_1{}^{\bullet}/\partial A_0 \neq 0$ holds as well as (3.1), then the expression for $A_0+\beta$ can be obtained from the second relation of (1.8) with (1.9) taken into account after which it can be inserted into the first relation of (1.8). This will give us an equation for γ, whose form will be similar to that of (3.3) for β. Rest of the computation remains analogous. In this case, the coefficients $B_{n/k}$ can be found independently from $A_{n/k}$ and the formulas defining them will exhibit a symmetry with the corresponding formulas for $A_{n/k}$. For example, $A_{\frac{1}{2}}{}^2$ proportional to Δ_1, while $B_{\frac{1}{2}}{}^2$ will be proportional to Δ_2.

4. As an example, we shall consider the Duffing problem in quasi-linear formulation. We have the harmonic equation

$$\ddot{x}+x=\mu\,(ax+bx^3+\nu\cos t+\lambda\sin t) \tag{4.1}$$

Let us write the fundamental amplitude equations

$$C_1\,(2\pi)=-\pi\,[\lambda+aB_0+{}^3/_4bB_0\,(A_0^2+B_0^2)]=0$$
$$C_1^{\bullet}\,(2\pi)=\pi\,[\nu+aA_0+{}^3/_4bA_0\,(A_0^2+B_0^2)]=0$$

We shall seek the periodic solutions corresponding to multiple roots of these equations. Constructing the determinant D_1 and equating it to zero, we obtain

$$D_1=\pi^2\,[a+{}^3/_4b\,(A_0^2+B_0^2)]\,[a+{}^9/_4b\,(A_0^2+B_0^2)]=0$$

Equating the first factor to zero we find that $\nu = 0$ and $\lambda = 0$. This case is of no special interest, as the system becomes self-contained.

Equating the second factor to zero, we obtain the following relation between the coefficients of (4.1):

$$16a^3+81b\,(\nu^2+\lambda^2)=0 \tag{4.2}$$

At the same time, the amplitudes A_0 and B_0 assume the values

$$A_0 = -\frac{3}{2}\frac{\nu}{a}, \qquad B_0 = -\frac{3}{2}\frac{\lambda}{a} \tag{4.3}$$

for which the determinant $D_2 \neq 0$. Consequently, the above roots of amplitude equations are double. Since $Q_2 \neq 0$, it follows that the parameters β and γ can be expanded in powers of $\mu^{\frac{1}{2}}$ and periodic solutions of (4.1) are represented by series of the form

$$x(t) = x_0(t) + \mu^{1/2} x_{1/2}(t) + \mu x_1(t) + \ldots \tag{4.4}$$

Let us find the coefficient $A_{\frac{1}{2}}$ from (3.15) and $B_{\frac{1}{2}}$ from the first equation of (3.16). Then, the function

$$x_{1/2}(t) = \mp(-96a)^{-1/2}(\nu \cos t + \lambda \sin t) \tag{4.5}$$

Hence, if $a < 0$ and (4.2) holds, then two real periodic solutions exist, which can be expanded in terms of $\mu^{\frac{1}{2}}$. Coefficients $x_0(t)$ and $x_{\frac{1}{2}}(t)$ have been found, and the coefficent $x_1(t)$ is

$$x_1(t) = \frac{7}{432}(\nu \cos t + \lambda \sin t) + \frac{27}{256}\frac{b}{a^3}[\nu(\nu^2 - 3\lambda^2)\cos 3t + \lambda(3\nu^2 - \lambda^2)\sin 3t] \tag{4.6}$$

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